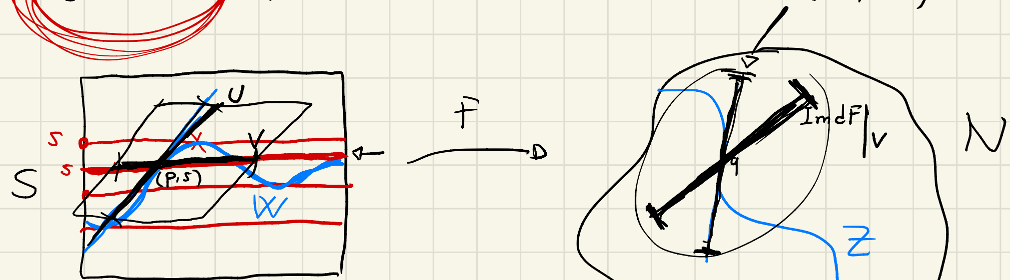
Teo (TRASVERSALITÀ DI THOM): $F: M \times S \to N \supseteq Z$

$$F \pitchfork Z \Rightarrow F_s \pitchfork Z \text{ per q.o. } s \in S \qquad F_s = F(\cdot, s)$$

dim



$\pi : M \times S \to S$

M

$\pi|_W$ $\qquad$ s val. reg per $\pi|_W$

$\Downarrow$

$F_s \pitchfork W$

$q = F(p,s)$

$\forall p \in W \cap M \times \{s\}$ $\quad F_s \pitchfork Z$ $\qquad F \pitchfork Z$

K-FORME DIFFERENZIALI

M varietà che può avere bordo.

Def: Una K-FORMA su M è $\omega \in \Gamma(\Lambda^K(M))$

$$\Lambda^k M = \bigsqcup_{p \in M} \left(\Lambda^k (T_p M) \right) \quad \text{fibrato}$$

$\forall p \in M \quad \omega(p) \in \Lambda^k(T_p M)$ cioè $\omega(p)(v_1, \dots, v_k) \in \mathbb{R}$ in modo antisimmetrico

$$\boxed{\Omega^k(M)} := \Gamma \Lambda^k(M)$$

$$\Omega^0(M) = \mathcal{C}^\infty(M) = \mathcal{C}^\infty(M, \mathbb{R})$$

$$\Omega^1(M) = \Gamma \Lambda^1(M) = \Gamma T^*(M)$$

Esempio: $f \in \mathcal{C}^\infty(M) = \Omega^0(M)$

$df \in \Gamma T^*(M) = \Omega^1(M)$

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M)$$

LINEARE

VEDREMO: $\Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M)$

$\Omega^k(M)$ sono $\mathcal{C}^\infty(M)$-modulo

$\omega, f \longrightarrow (f\omega)(p) = f(p)\omega(p)$

PRODOTTO WEDGE: $\wedge$

$\omega \in \Omega^k(M) \quad \eta \in \Omega^h(M) \quad \omega \wedge \eta \in \Omega^{k+h}(M)$

$$(\omega\wedge\eta)(p) = \omega(p)\wedge\eta(p)$$

Questa operazione è associativa, anticommutativa

$$\omega\wedge\eta = (-1)^{k\cdot h}\,\eta\wedge\omega \quad \leftarrow \text{vero } \forall p$$

è liscia perché può essere scritta in coordinate

ω 1-forma $\Rightarrow$ $\omega\wedge\omega = 0$

$$\to \boxed{\Omega^*(M) := \bigoplus_{k\geq 0} \Omega^k(M)}$$

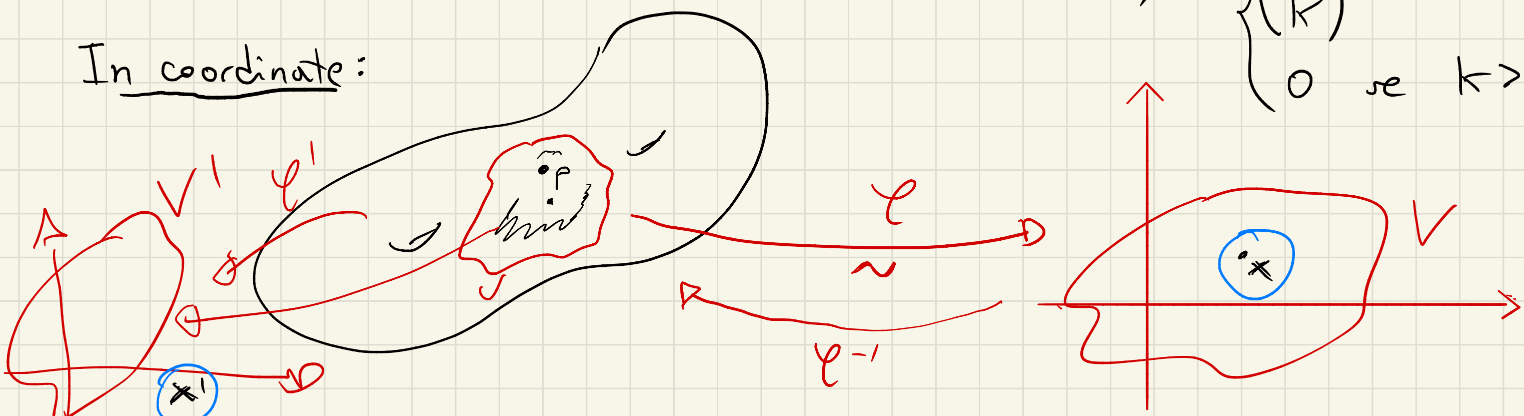
somma finita $0\leq k\leq n$

$$\Omega^k(M) = \{0\} \text{ se } k>n$$

Ricordiamo che

$$\operatorname{rk} \Lambda^k(M) = \begin{cases} \binom{n}{k} & \text{se } k\leq n \\ 0 & \text{se } k>n \end{cases}$$

In coordinate:



$\omega \in \Omega^k(M)$ si restringe a $\omega \in \Omega^k(U) \xrightarrow[\varphi_*]{\sim} \Omega^k(V)$

$\varphi_*(\omega) \in \boxed{\Omega^k(V)}$ $\quad V \subseteq \mathbb{R}^n$

La chiamo sempre ω

$x \in V \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad T_x \mathbb{R}^n = \mathbb{R}^n$

$\omega(x) \in \Lambda^k(\mathbb{R}^n)$ Base per $\Lambda^k(\mathbb{R}^n)$ è data da

$\binom{n}{k} \quad \{ e^{i_1} \wedge \dots \wedge e^{i_k} \}$ $\quad \{e^1, \dots, e^n\} \in (\mathbb{R}^n)^*$

base duale

$0 < i_1 < \dots < i_k \leq n$ $\quad$ della base canonica $\{e_1, \dots, e_n\}$ $\mathbb{R}^n$

$$\omega(x) = \sum_{i_1 < \dots < i_k} f_{i_1, \dots, i_k}(x)\, e^{i_1} \wedge \dots \wedge e^{i_k}$$

$e_i \dashrightarrow \dfrac{\partial}{\partial x^i}$ $\qquad$ $e^i \dashrightarrow dx^i$ $\qquad$ $x^i : \mathbb{R}^n \to \mathbb{R}$

$e^i = dx^i$ $\qquad\qquad\qquad$ $x \mapsto x^i$

Notazioni che sono utili quando cambiamo coordinate

$$\omega(x) = \sum_{i_1 < \dots < i_k} f_{i_1 \dots i_k}(x)\, dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

$$\int \dots \, dx\, dy \qquad \begin{aligned} x &= \rho \cos\vartheta \\ y &= \rho \operatorname{sen}\vartheta \end{aligned} \qquad \begin{aligned} dx = {} & \cos\vartheta \, d\rho \\ & - \rho \operatorname{sen}\vartheta \, d\vartheta \end{aligned}$$

Con altre coordinate $\quad \bar{x} \dashleftarrow x$

$$\boxed{d\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^j} dx^j} \qquad \boxed{\frac{\partial}{\partial \bar{x}^i} = \frac{\partial x^j}{\partial \bar{x}^i} \frac{\partial}{\partial x^j}}$$

Esempi: $\quad f \in \mathcal{C}^\infty(V) \qquad V \subseteq \mathbb{R}^n$

$$\boxed{df = \frac{\partial f}{\partial x^1} dx^1 + \ldots + \frac{\partial f}{\partial x^n} dx^n}$$

Oss: $\omega \in \Omega^n(\mathbb{R}^n)$ $\qquad \omega = f(x)\, dx^1 \wedge \ldots \wedge dx^n$

Cambiando coordinate otteniamo $\qquad x \longrightarrow \bar{x}$

$$\omega = f(\bar{x}) \det\left(\frac{\partial x}{\partial \bar{x}}\right) d\bar{x}^1 \wedge \ldots \wedge d\bar{x}^n$$

è molto simile a formula cambio coord. in $\int$

$$\boxed{dx^1 \wedge \ldots \wedge dx^n = \det\left(\frac{\partial x}{\partial \bar{x}}\right) d\bar{x}^1 \wedge \ldots \wedge d\bar{x}^n}$$

$v_1, \ldots, v_n$ basi di V $\longrightarrow$ $v^1 \wedge \ldots \wedge v^n \in \Lambda^n(V)$

$w_1, \ldots, w_n$ $\qquad w^1 \wedge \ldots \wedge w^n \in \Lambda^n(V)$

$$w^1 \wedge \ldots \wedge w^n = \det(\quad)\, v^1 \wedge \ldots \wedge v^n$$

$$V \xrightarrow{\sim} W \qquad\qquad T^k_0(V) \xrightarrow{\sim} T^k_0(W)$$

$\varphi : M \to N$ diffeo $\qquad \omega \in \Omega^k(M)$

$\dashrightarrow \varphi_* \omega \in \Omega^k(N)$

$q \in N$

$$\underbrace{(\varphi_* \omega)}\underbrace{(q)}\underbrace{(v_1, \dots, v_k)}_{v_i \in T_q N} = \omega(\varphi^{-1} q)(w_1, \dots, w_k)$$

$$w_i = d\varphi^{-1}_q (v_i)$$

PULL-BACK

Def: $\boxed{f : M \to N}$ qualsiasi liscia $\qquad \boxed{df_p : T_p M \to T_{f(p)} N}$

$\omega \in \Omega^k(N) \dashrightarrow f^* \omega \in \Omega^k(M)$

$$(f^* \omega)(p)(v_1, \dots, v_k) = \omega(f(p))(df_p(v_1), \dots, df_p(v_k))$$

$f : M \to N \dashrightarrow f^* : \Omega^k(N) \to \Omega^k(M)$

In carte: $\omega \in \Omega^k(M)$ $\qquad \eta \in \Omega^h(M)$

$$\omega = \sum_{i_1 < \dots < i_k} f_{i_1 \dots i_k}(x)\, \underline{dx^{i_1} \wedge \dots \wedge dx^{i_k}}$$

$$\boxed{= \sum_I f_I \, dx^I}$$

MULTIINDICE

$I = (i_1, \dots, i_k)$

$i_1 < \dots < i_k$

$$\eta = \sum_J g_J \, dx^J$$

$J = (j_1, \dots, j_h)$

$$\omega \wedge \eta = \left(\sum_I f_I \, dx^I\right) \wedge \left(\sum_J g_J \, dx^J\right)$$

$$\sum_{I,J} f_I g_J \, \underline{dx^I \wedge dx^J}$$

$$\| $$

$$dx^{j_1} \wedge \dots \dots \wedge dx^{j_{h+k}}$$

$$\underline{j_1 < \dots < j_{h+k}}$$

$(dx^1 \wedge dx^3) \wedge (dx^2 \wedge dx^5)$

$dx^1 \wedge dx^3 \wedge dx^2 \wedge dx^5$

$\Downarrow$

$- dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^5$

$dx^i \wedge dx^j = - dx^j \wedge dx^i$

$dx^i \wedge dx^i = 0$

$f: M \to N$ $\qquad$ $f^*: \Omega^k(N) \to \Omega^k(M)$

Prop: $f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta)$

fatto puntuale conseguenza della definizione naturale di $\wedge$

CONTRAZIONI

M $\qquad$ $X \in \mathfrak{X}(M)$ $\qquad$ $\omega \in \Omega^k(M)$ $\qquad$ $\iota_X(\omega) \in \Omega^{k-1}(M)$

$$\iota_X(\omega)(p) = \iota_{X(p)} \omega(p)$$

$$\iota_X(\omega)(p)(v_1, \dots, v_{k-1}) = \omega(p) = (X(p), v_1, \dots, v_{k-1})$$

<u>INTEGRAZIONE</u>

$\omega \in \Omega^n_c(V)$ $\qquad$ $V \subseteq \mathbb{R}^n$ $\qquad$ $\omega = f(x)\, dx^1 \wedge \dots \wedge dx^n$

$$\int_V \omega := \int_V f$$

Lebesgue

$\omega \in \Omega^n(M)$ $\qquad \mathrm{supp}(\omega) = \overline{\{ p \in M \mid \omega(p) \neq 0 \}}$

$$\underbrace{\Omega^k_c(M)}_{\{k\text{-forme a supporto cpt}\}} \subseteq \Omega^k(M)$$

<u>Oss</u>: $V \subseteq \mathbb{R}^n$ $\quad V' \subseteq \mathbb{R}^n$ $\quad \varphi : V \to V'$ diffeo$^+$ $(\det(d\varphi_x) > 0 \ \forall x \in V)$

$\omega \in \Omega^n(V)$

$$\boxed{\int_V \omega = \int_{V'} \varphi_*(\omega)}$$

$$\varphi_* = (\varphi^*)^{-1} = (\varphi^{-1})^*$$

$$\int_V f = \int_{V'} |\det d\varphi| f = \int_{V'} \det d\varphi \, f = \int_{V'} \varphi_*(\omega)$$

$$\omega = f dx^1 \wedge - \wedge dx^n$$

<u>Def</u>: M orientata $\quad \omega \in \Omega^n_c(M)$

$$\boxed{\int_M \omega = ?}$$

Supponiamo $\mathrm{supp}\, \omega \subseteq U \subseteq M$

$U \xrightarrow{\varphi} V \subseteq \mathbb{R}^n$ φ pres. or.

In questo caso definisco $\boxed{\int_M \omega = \int_V \varphi_* \omega}$

φ e φ' pres. or.

$\Downarrow$

$\varphi' \circ \varphi^{-1}$:

$\varphi(U \cap U') \to \varphi'(U \cap U')$

pres. or.

Buona def?

$\operatorname{supp} \omega \subseteq U' \xrightarrow{\varphi'} V'$

$$\int_M \omega = \int_{V'} \varphi'_* \omega$$

Def: In generale $\operatorname{supp} \omega$ è cpt.

$\{\varphi_i : U_i \to V_i\}$ atlante orientato $\dashrightarrow$ $\{\rho_i\}$ partiz. 1.

$\operatorname{supp} \rho_i \subseteq U_i$

$$\to \omega = \left(\sum \rho_i\right) \omega = \sum_{i \in I} \rho_i \omega$$

$$= \sum_{i \in I} \omega_i$$

$\boxed{\omega_i = \rho_i \omega}$ $\to \operatorname{supp} \omega_i \subseteq U_i$

$\rho_i \in \mathcal{C}^\infty(M)$

$$\int_M \omega := \sum_{i\in I} \int_M \omega_i \quad \text{ha senso} \qquad \Big| \quad \text{supp } \omega \text{ è cpt} \Rightarrow \underbrace{\omega = \sum_{i\in I} \omega_i}\ \text{è finita}$$

<u>Oss</u>: Non dip. da niente:

$$\{\varphi_j' : U_j' \to V_j'\} \qquad \{\rho_j'\}$$

$$\text{II def} \int_M \omega = \sum_{j\in J} \int_M \omega_j' = \sum_j \int_M \underbrace{\rho_j' \omega}$$

$$= \sum_j \int_M \Big(\sum_{i\in I} \rho_i\Big) \rho_j' \omega$$

$$= \sum_j \int_M \sum_i \underbrace{(\rho_i \rho_j' \omega)} = \sum_j \sum_i \int_M \rho_i \rho_j' \omega$$

$$= \sum_{i,j} \int_M \rho_i \rho_j' \omega = \dots = \int_M \omega \quad \text{I def}$$

$\varphi : M \to N$ diffeo

$d\varphi_p : T_p M \xrightarrow{\sim} T_{\varphi(p)} N$

$\varphi_* : \Omega^k(M) \to \Omega^k(N)$